Monte Carlo method for multiparameter estimation in coupled chaotic systems

Inés P. Mariño[1,*] and Joaquín Míguez[2,†]
[1]*Departamento de Física, Universidad Rey Juan Carlos, 28933 Móstoles, Madrid, Spain*
[2]*Departamento de Teoría de la Señal y Comunicaciones, Universidad Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain*
(Received 29 July 2007; published 16 November 2007)

We address the problem of estimating multiple parameters of a chaotic dynamical model from the observation of a scalar time series. We assume that the series is produced by a chaotic system with the same functional form as the model, so that synchronization between the two systems can be achieved by an adequate coupling. In this scenario, we propose an efficient Monte Carlo optimization algorithm that iteratively updates the model parameters in order to minimize the synchronization error. As an example, we apply it to jointly estimate the three static parameters of a chaotic Lorenz system.

DOI: 10.1103/PhysRevE.76.057203 PACS number(s): 05.45.Xt, 05.45.Gg, 05.45.Pq, 05.45.Tp

I. INTRODUCTION

An important topic of research in time series analysis of nonlinear systems is the estimation of the parameters which are needed in a numerical model in order to make it follow the dynamics of the original system, from which a scalar series of observations can be collected. Provided that the functional form of the model is accurate enough, these estimates can subsequently be used to track the dynamic system state. The problem of parameter estimation can be tackled in different ways, e.g., using multiple shooting methods [1,2] or some statistical procedures based on time discretization and other approximations [3–7]. However, these methods involve the solution of high-dimensional minimization problems, since not only the unknown parameters but also the initial values of the trajectory segments between the sampling times need to be estimated [2,8].

Recently, several authors have suggested to take advantage of the synchronization techniques of coupled chaotic systems and turn them into accurate parameter estimation methods [8–19]. This approach is appealing because the only unknowns are the parameters to be estimated, hence only low-dimensional optimization problems need to be tackled.

In most estimation methods, the unknown model parameters are handled as dynamic variables, with associated differential equations that must be designed (in a nontrivial way) to ensure convergence to the desired values [8–14]. Recently, however, Sakaguchi has proposed, in [19], a method to adjust the model parameter values using chaos synchronization and a Monte Carlo procedure which is conceptually simple, effective, and for which the design of additional differential equations is not necessary. A direct numerical optimization of the error between the model and the scalar time series is carried out instead.

In this paper, we focus on this scenario and propose a parameter estimation method that takes advantage of chaos synchronization and is based on a more efficient Monte Carlo optimization procedure, known as accelerated random search (ARS) [20]. The algorithm performs an iterative update of the model parameters in order to minimize the synchronization error between the available observations, collected from the system of interest, and the model dynamic variables. We will show that the ARS-based estimation algorithm attains more accurate results than Sakaguchi's technique, while remaining equally simple from a conceptual viewpoint and having the same computational complexity.

The rest of the paper is organized as follows. In Sec. II, we describe the problem of multiparameter estimation for coupled dynamical systems. Section III contains a detailed description of the proposed estimation method. An application example, the estimation of the three static parameters of a chaotic Lorenz system, is presented in Sec. IV. Finally, Sec. V is devoted to the conclusions.

II. PROBLEM STATEMENT

We first introduce the mathematical model and notation to be used throughout the rest of the paper. Let

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{p}) \tag{1}$$

represent the *primary* chaotic system with state variables $\mathbf{x}=(x_1, \ldots, x_n) \in \mathbb{R}^n$ and unknown parameters $\mathbf{p}=(p_1, \ldots, p_m) \in \mathbb{R}^m$. If the functional form of (1) is known, we can build a model, termed *secondary* system in the sequel, as

$$\dot{\mathbf{y}} = \mathbf{g}(\mathbf{y}, \mathbf{q}) + D\mathbf{s}_i(\mathbf{x} - \mathbf{y}), \tag{2}$$

with $\mathbf{g}$ identical to $\mathbf{f}$, and where $\mathbf{y}=(y_1, \ldots, y_n) \in \mathbb{R}^n$ is the time-varying vector that contains the model state variables, $\mathbf{q}=(q_1, \ldots, q_m) \in \mathbb{R}^m$ is the adjustable parameter vector, D is a coupling coefficient, and $\mathbf{s}_i: \mathbb{R}^n \to \mathbb{R}^n$ is a vector function that selects the ith element of its argument, i.e., $\mathbf{s}_i(\mathbf{x}-\mathbf{y}) = (0, \ldots, 0, x_i - y_i, 0, \ldots, 0)$. The definition of the latter function implies that coupling is performed only through the scalar time series x_i from the primary system.

Since we assume $\mathbf{g}=\mathbf{f}$, when the secondary parameter vector, $\mathbf{q}$, coincides with the primary parameter vector, $\mathbf{p}$, the state variables $\mathbf{y}$ synchronizes with $\mathbf{x}$ for $D > D_c$, where D_c is a coupling threshold [19]. On the contrary, if $\mathbf{q} \neq \mathbf{p}$ then

*ines.perez@urjc.es
†joaquin.miguez@uc3m.es

complete synchronization cannot occur. However, the difference $\|\mathbf{y}-\mathbf{x}\|$ is expected to be small whenever the difference of the two parameter vectors is small and D is sufficiently large. Therefore, the objective of a parameter estimation method based on synchronization is to calculate a value $\hat{\mathbf{q}}$ such that $\|\mathbf{y}-\mathbf{x}\| \approx 0$, since the latter implies $\hat{\mathbf{q}} \approx \mathbf{p}$.

III. PARAMETER ESTIMATION METHOD

Assuming that the scalar time series x_i from the primary system is observed during the time interval $(0, T_o)$, the value of the parameters in the secondary system can be calculated as the solution to the optimization problem [19]

$$\hat{\mathbf{q}} = \arg \min_{\mathbf{q}} \{U(\mathbf{q})\}, \tag{3}$$

where the cost function $U(\mathbf{q}) = \int_0^{T_o} |x_i - y_i|^2 dt$ is a quantitative representation of the synchronization error between the primary and the secondary systems.

In [19], a simple Monte Carlo optimization method is proposed to solve problem (3). While effective, this technique can show a very slow convergence. More efficient algorithms, with similar complexity, can be designed, however. Here, we introduce an iterative optimization technique based on the ARS methodology of [20]. The proposed algorithm can be outlined as follows:

(1) *Initialization.* Choose initial parameter values $\hat{\mathbf{q}}(0) = [\hat{q}_1(0), \hat{q}_2(0), \ldots, \hat{q}_m(0)]$, maximum and minumum "search variances," $\sigma^2_{\max}$ and $\sigma^2_{\min}$, respectively, and set $\sigma^2(1) = \sigma^2_{\max}$. Also choose a *contraction factor*, $\kappa > 1$. Using the initial parameter vector, $\hat{\mathbf{q}}(0)$, evaluate the associated cost $U[\hat{\mathbf{q}}(0)]$.

(2) *Iterative step.* Let $N(\mu, \sigma^2)$ denote the Gaussian probability distribution with mean μ and variance σ^2. Given the $(n-1)$th parameter estimates, $\hat{\mathbf{q}}(n-1)$, with associated cost $U[\hat{\mathbf{q}}(n-1)]$, and the search variance value $\sigma^2(n)$, proceed with the following steps.

(a) Randomly draw new parameter values

$$\tilde{q}_i \sim N[\hat{q}_i(n-1), \sigma^2(n)] \quad \text{for } i = 1, \ldots, m,$$

and build the auxiliary parameter vector $\tilde{\mathbf{q}} = (\tilde{q}_1, \ldots, \tilde{q}_m)$.

(b) Compute the associated cost, $U(\tilde{\mathbf{q}})$.

(c) If $U(\tilde{\mathbf{q}}) < U[\hat{\mathbf{q}}(n-1)]$, then update the parameter estimate, its associated cost, and the search variance value,

$$\hat{\mathbf{q}}(n) = \tilde{\mathbf{q}},$$

$$U[\hat{\mathbf{q}}(n)] = U(\tilde{\mathbf{q}}),$$

$$\sigma^2(n+1) = \sigma^2_{\max}.$$

Else, if $U(\tilde{\mathbf{q}}) \geq U[\hat{\mathbf{q}}(n-1)]$, then preserve the old parameter values,

$$\hat{\mathbf{q}}(n) = \hat{\mathbf{q}}(n-1),$$

$$U[\hat{\mathbf{q}}(n)] = U[\hat{\mathbf{q}}(n-1)],$$

and contract the search variance,

$$\sigma^2(n+1) = \sigma^2(n)/\kappa.$$

(d) If $\sigma^2(n+1) < \sigma^2_{\min}$, then set $\sigma^2(n+1) = \sigma^2_{\max}$.

The algorithm can be stopped after a fixed number of iterations or when $U[\hat{\mathbf{q}}(n)] < \epsilon$, for some prescribed $\epsilon > 0$. Note that the time evolution of the secondary system state, $\mathbf{y}$, must be recalculated at each iteration, since a new candidate vector, $\tilde{\mathbf{q}}$, is drawn each time

Intuitively, the proposed algorithm attempts to improve the parameter estimates by adaptively constraining the search variance around them. In other words, if a given parameter vector has smaller cost than the proposed candidates during several successive iterations of the algorithm, i.e.,

$$\hat{\mathbf{q}}(n) = \hat{\mathbf{q}}(n+1) = \cdots = \hat{\mathbf{q}}(n+k) = \mathbf{q}',$$

then we can argue that $\mathbf{q}'$ is a good estimate, and hence it should be close to the optimum parameter vector. In that case, any better solution can be expected to be close to $\mathbf{q}'$ and, therefore, it makes sense that the search variance becomes very small (as several contractions are applied successively). When either we find a new vector with a smaller cost or the search variance moves below a given threshold ($\sigma^2_{\min}$), we reset the search variance to its maximum value ($\sigma^2_{\max}$) in order to give the algorithm a chance to find good solutions that may lie far away from the current estimate. This is important for the optimization of cost functions with multiple (and possibly sparse) minima, to avoid that the algorithm becomes stuck in a local optimum.

Compared to the original ARS algorithm in [20], the proposed method uses Gaussian distributions with an adaptive search variance in order to propose parameter estimates. In [20], a uniform distribution on intervals with adaptive length is used instead. We conjecture that when the cost function, $U(\mathbf{q})$, has many local minima, it can be hard to find adequate maximum and minimum interval lengths that lead to fast convergence of the original ARS method. The proposed Gaussian ARS (GARS), on the other hand, may exhibit faster convergence to the global optimum because the domain of the proposed parameter values is not deterministically bounded.

Comparing the GARS method with the technique proposed by Sakaguchi in [19], we see that the latter does not adaptively contract or expand the search variance. As a consequence, estimation with Sakaguchi's method is less accurate than with the GARS, as will be numerically shown in the next section.

IV. APPLICATION TO THE LORENZ SYSTEM

Let us illustrate the application of the proposed method by way of an example that involves the Lorenz system. Thus, we assume the primary system

$$\dot{x}_1 = -p_1(x_1 - x_2),$$

$$\dot{x}_2 = p_2 x_1 - x_2 - x_1 x_3,$$

$$\dot{x}_3 = -p_3 x_3 + x_1 x_2, \tag{4}$$

where $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ forms the state space. The fixed parameters in Eq. (5) are $\mathbf{p} = (p_1, p_2, p_3) = (10, \frac{8}{3}, 28)$. The secondary system is

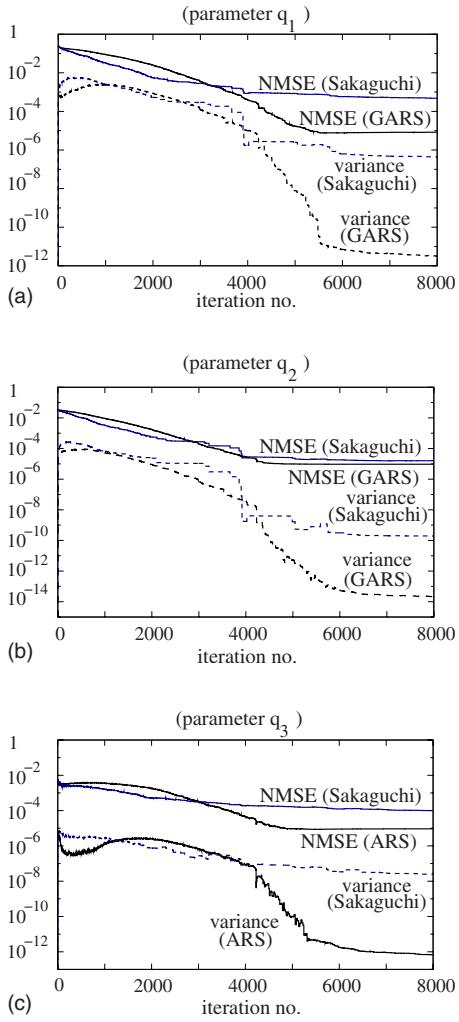

FIG. 1. (Color online) Normalized mean square error $[E_i(n)$, solid lines] and its variance $[\sigma_i^2(n)$, dashed lines] in the estimation of p_1 (upper plot), p_2 (middle plot), and p_3 (lower plot) versus the iteration number. All three NMSEs have been averaged over 200 independent simulation trials.

$$\dot{y}_1 = -q_1(y_1 - y_2) + D(x_1(t) - y_1),$$

$$\dot{y}_2 = q_2 y_1 - y_2 - y_1 y_3,$$

$$\dot{y}_3 = -q_3 y_3 + y_1 y_2, \tag{5}$$

where $\mathbf{y}=(y_1,y_2,y_3)\in\mathbb{R}^3$ contains the state variables, $\mathbf{q}=(q_1,q_2,q_3)\in\mathbb{R}^3$, and x_1 is the time series from the primary system, which is observed from time 0 to time $T_o=50$ time units (t.u.). Many types of coupling terms between systems (4) and (5) that lead to chaos synchronization are possible, but we have assumed the above form of coupling for the sake of simplicity. Direct numerical simulations show that complete synchronization, i.e., $y_1=x_1$, occurs for $D>D_c=7.95$ when $\mathbf{q}=\mathbf{p}$. For this reason, we have set $D=9$ for the numerical simulations in this section.

We have numerically integrated the two chaotic systems using the fourth-order Runge-Kutta method with a time step

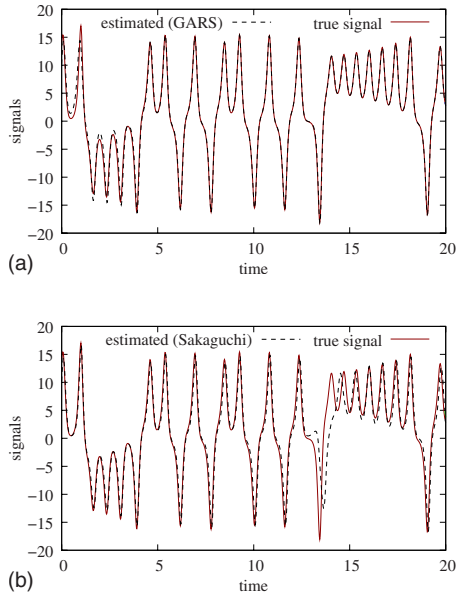

FIG. 2. (Color online) Example of synchronization between the first dynamic variables of the primary and secondary systems, x_1 and y_1, respectively, when the GARS (upper plot) and Sakaguchi's technique (lower plot) have been used to compute the parameters of the secondary system.

$T_s=10^{-3}$ t.u. In this way, we obtain equally spaced samples of both state vectors, $\mathbf{x}$ and $\mathbf{y}$, at time instants $t=nT_s$, for $n\in\mathbb{N}$, and we can approximate the integral in the cost function U as

$$U(\mathbf{q}) = \int_0^{T_o} (y_1 - x_1)^2 dt \approx T_s \sum_{n=0}^{N-1} [y_1(nT_s) - x_1(nT_s)]^2,$$

where $N=\lfloor T_o/T_s \rfloor$.

Within this simulation setup, we have applied both the proposed GARS algorithm, with parameters $\sigma^2_{\max}=1$, $\sigma^2_{\min}=10^{-5}$, and $\kappa=1.5$, and the method by Sakaguchi, with fixed search variance $\sigma^2=1$. Both Monte Carlo optimization algorithms are iterated 8000 times, and the secondary system must be integrated for each iteration. The initialization of the parameter vector $\mathbf{q}$ is also the same for both techniques, namely $\hat{\mathbf{q}}(0)=\mathbf{p}+(-5,0.2,5)$.

We first consider the normalized mean square error (NMSE) attained in the estimation of each parameter, which we have approximated by averaging over 200 independent simulations. Specifically, let $\hat{q}_i(n,k)$ be the estimate of p_i, $i=1,2,3$, obtained in the kth simulation after n iterations of the algorithm. The associated normalized square error is $E_i(n,k)=\{[p_i-\hat{q}_i(n,k)]/p_i\}^2$, $i\in\{1,2,3\}$, and the NMSE is the average

$$E_i(n) = \frac{1}{200}\sum_{k=1}^{200} E_i(n,k), \quad i\in\{1,2,3\}. \tag{6}$$

We have additionally calculated the sample variance of Eq. (6), namely $\sigma_i^2(n)=\frac{1}{200}\sum_{k=1}^{200}[E_i(n,k)-E_i(n)]^2$, $i\in\{1,2,3\}$.

Figure 1 shows the obtained results. The upper plot shows the NMSE and its variance for parameter p_1, i.e., curves $E_1(n)$ and $\sigma_1^2(n)$, the middle plot shows the NMSE, $E_2(n)$, and its variance, $\sigma_2^2(n)$, for parameter p_2, and the lower plot shows the corresponding NMSE, $E_3(n)$, and variance, $\sigma_3^2(n)$, for parameter p_3. We observe that the error is very similar in the three cases for the GARS algorithm, $E_i(8000) \approx 10^{-5}$ for $i=1,2,3$, and it always outperforms Sakaguchi's method, especially when estimating p_1 and p_3, while the advantage for the estimation of p_2 is not so large. In the first case (p_1 and p_3) the NMSE achieved with the GARS technique is more than one order of magnitude smaller, while in the second case (p_2), the NMSE provided by the GARS algorithm is approximately one-half of the error obtained via Sakaguchi's technique. Furthermore, we can also observe that the variance of the NMSE in the simulations is much smaller for the GARS method.

In agreement with the previous results, the synchronization between the primary and secondary systems is more accurate, in the average, when the GARS estimation algorithm is used to compute the desired values of $\hat{\mathbf{q}}(n)$. As an example, Fig. 2 shows a typical realization of the state variable y_1 (together with the observed signal x_1 from the primary system), collected from the last iteration of both the GARS method (upper plot) and Sakaguchi's algorithm (lower plot).

V. CONCLUSIONS

We have addressed the problem of estimating the multiple static parameters of a chaotic system from the observation of a scalar time series. Our approach is based on the synchronization properties of coupled chaotic systems. For the numerical computation of the parameter estimates, we have proposed a Monte Carlo optimization algorithm termed Gaussian accelerated random search. The GARS is an iterative method to find the parameter values that minimize the synchronization error between the observed time series and a model system. We have successfully applied the proposed technique to the joint estimation of the three parameters of a Lorenz chaotic system from which only one dynamic variable can be observed.

ACKNOWLEDGMENTS

One of the authors (I.P.M) acknowledges financial support by Universidad Rey Juan Carlos and Comunidad de Madrid under Project No. URJC-CM2006-CET-0643 and by the Spanish Ministry of Science and Education under Project No. FIS2006-08525. One of the authors (J.M.) acknowledges the support of the Ministry of Science and Education of Spain (MONIN, Project No. TEC2006-13514-C02-01/TCM) and Comunidad de Madrid (PRO-MULTIDIS, Project No. S0505/TIC/0223).

[1] A. Ghosh, V. R. Kumar, and B. D. Kulkarni, Phys. Rev. E **64**, 056222 (2001).
[2] E. Baake, M. Baake, H. G. Bock, and K. M. Briggs, Phys. Rev. A **45**, 5524 (1992).
[3] V. Petridis, E. Paterakis, and A. Kehagias, IEEE Trans. Neural Netw. **9**, 862 (1998).
[4] J. Timmer, Chaos, Solitons Fractals **11**, 2571 (2000).
[5] H. Singer, J. Comput. Graph. Statist. **11**, 972 (2002).
[6] A. Sitz, U. Schwarz, J. Kurths, and H. U. Voss, Phys. Rev. E **66**, 016210 (2002).
[7] V. F. Pisarenko and D. Sornette, Phys. Rev. E **69**, 036122 (2004).
[8] U. Parlitz, L. Junge, and L. Kocarev, Phys. Rev. E **54**, 6253 (1996).
[9] U. Parlitz, Phys. Rev. Lett. **76**, 1232 (1996).
[10] C. Zhou and C.-H. Lai, Phys. Rev. E **59**, 6629 (1999).
[11] A. Maybhate and R. E. Amritkar, Phys. Rev. E **59**, 284 (1999).
[12] A. d'Anjou, C. Sarasola, F. J. Torrealdea, R. Orduna, and M. Grana, Phys. Rev. E **63**, 046213 (2001).
[13] R. Konnur, Phys. Rev. E **67**, 027204 (2003).
[14] D. Huang, Phys. Rev. E **69**, 067201 (2004).
[15] U. S. Freitas, E. E. N. Macau, and C. Grebogi, Phys. Rev. E **71**, 047203 (2005).
[16] I. P. Mariño and J. Míguez, Phys. Rev. E **72**, 057202 (2005).
[17] I. P. Mariño and J. Míguez, Phys. Lett. A **351**, 262 (2006).
[18] C. Tao, Y. Zhang, and J. J. Jiang, Phys. Rev. E **76**, 016209 (2007).
[19] H. Sakaguchi, Phys. Rev. E **65**, 027201 (2002).
[20] M. J. Appel, R. Labarre, and D. Radulovic, SIAM J. Optim. **14**, 708 (2003).